Chaos in Robertson-Walker Cosmology

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Abstract

Chaos in Robertson-Walker cosmological models where gravity is coupled to one or more scalar fields has been studied by a few authors, mostly using numerical simulations. In this paper we begin a systematic study of the analytical aspect. We consider one conformally coupled scalar field and, using the fact that the model is integrable when the field is massless, we show in detail how homoclinic chaos arises for nonzero masses using a perturbative method.

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I. Introduction

In Ref. 1, Calzetta and El Hasi presented an argument, supported by numerical evidence, for the appearance of chaotic behaviour in a spatially closed Robertson-Walker (RW) cosmological model, filled with a conformally coupled massive scalar field. Because cosmological models of this type are practically the simplest possible ones, but give in some sense a rough, smoothed out indication of the average properties of a more realistic cosmology, they have frequently been used as a first testing ground on which to try out approaches to both classical and quantum problems in cosmology. In addition, because of their kinematical simplicity, one would expect the dynamics of these models to be equally simple. It thus may come somewhat as a surprise that chaos should appear at this level, although it is known that it can appear in a nonlinear dynamical system with two degrees of fredom, and one may suspect that it is in fact generic for relativistic systems.

The study of chaos in relativistic cosmology began more than twenty years ago with the study by the Russian school[2] and Misner's group[3] on diagonal Bianchi type IX (mixmaster) vacuum cosmologies. This research is, by itself, very interesting and still active,[4] but the hopes initially placed in these models, essentially the solution of the horizon problem from the mixing effect associated with chaos, could not be sustained, and the Bianchi IX cosmologies have been mostly used as toy models, rather than first approximations from which to extract estimates of physical quantities. On the other hand, although the question of which RW model best approximates our inhomogeneous, anisotropic universe is not yet settled, there is agreement on the relevance of RW models both for classical and quantum cosmology,[5] and physical predictions are often made based on them. Therefore the presence of chaos, in this kind of cosmologies, enlarges the avenues of research in fundamental subjects such as the arrow of time and the issue of irreversibility in the evolution of the universe, or the transition from the quantum regime to the classical one, with the corresponding appearance of decoherence and correlations and related phenomena like particle creation, and has been used in considerations related to physical phenomena in inflationary universes.[6,7,8]

In their work, Calzetta and El Hasi used mostly a numerical method to investigate the chaotic behaviour of the model, but also sketched an analytical, perturbative method by which they considered the model as a perturbation of the integrable one obtained with a vanishing mass for the scalar field, and split the Hamiltonian of the system into an integrable part and a coupling term. With a perturbative argument commonly used in the treatment of near-integrable systems, based on the Melnikov criterion for homoclinic chaos

and Chirikov's resonance overlap criterion,[9] they argued qualitatively that the KAM tori of the integrable part are destroyed and replaced by stochastic layers under the effect of the perturbation. The resulting strong indications of chaotic behaviour were confirmed by the numerical analysis, which allowed them to go beyond perturbation theory.

The generic transition to chaos in a perturbed integrable Hamiltonian system ("soft chaos") can be seen as taking place in two steps.[9,10] In the first one, a perturbation of the Hamiltonian, with a nonvanishing Fourier component in resonance with one of the tori of the unperturbed system, modifies the dynamics in a neighborhood of this torus to give a new integrable system, which however has heteroclinic orbits even if before there were none. These orbits are separatrices, which act as seeds of chaos when in the second step one takes into account the effect of the remaining components of the perturbation on the new integrable dynamics; it is here that one may use the Melnikov method to determine whether in effect the dynamics becomes chaotic. What this method provides is a topological criterion for detecting a chaotic "tangle" in the orbits near the separatrices, and is therefore free of the coordinate ambiguities that have arisen, e.g., in the study of the chaotic nature of Bianchi models by other methods; for a quick introduction to the Melnikov method see, e.g., Refs. 11 or 12, and Ref. 13 for a more technical exposition.

The goal of this paper is to continue the work in Ref. 1 by starting to examine in detail the analytical treatment of the model; in future work, we will systematically extend our work to more general models. We will use a different perturbative expansion of the Hamiltonian, and show the onset of chaotic behavior by an explicit computation of the Melnikov integral. Other recent work on chaos in Robertson-Walker models coupled to scalar fields has focused on their application to inflation,[7] again using mostly numerical methods, and an analysis based on the Painlevé analysis of differential equations,[14] which confirmed the non-integability of the models. The main advantage of our work is the amount of information it can give us regarding the resonances responsible for the onset of chaos and the chaotic region of phase space.

The paper is organized as follows: in section II, we set up the perturbative method we will use, by choosing appropriate variables to describe the cosmological model and separating the corresponding Hamiltonian into an "unperturbed" term and a "perturbation;" in section III we analyze the first step above for our example, i.e., we discuss the local dynamics in the vicinity of one of the resonant tori, while in section IV we develop the second step, discussing the effect of other perturbation terms on the local dynamics, which becomes chaotic as Melnikov's method shows; section V contains concluding remarks on possible future developments.

II. Dynamical variables and Hamiltonian: setup for the perturbative treatment

Let us consider the closed Robertson-Walker (RW) metric for a homogeneous and isotropic universe, written in the form

$$\mathrm{d}s^2 = a^2(t)[-\mathrm{d}t^2 + \mathrm{d}\chi^2 + \sin^2\!\chi(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\varphi^2)]\;, \tag{2.1}$$

where $0 \le \varphi \le 2\pi$, $0 \le \theta \le \pi$, $0 \le \chi \le \pi$ are the angular coordinates on S^3, and t is the usual "conformal time" (see, e.g., Ref. 15). We shall consider only models possessing a cosmic singularity $a = 0$, chosen for convenience to occur at $t = 0$. We shall also assume that after the big crunch a new cosmological cycle begins, but a changes sign, in such a way that the evolution is smooth, and this process is repeated an infinite number of times. This well-known possibility of extending the evolution beyond the $a = 0$ singularity is an artifact of the symmetry possessed by the RW models (we have no grounds for believing that it can be generalized), but it is convenient for us because it allows us to assume that time runs up to $t = +\infty$, and thus to use techniques for detecting chaos, including its very definition, that would otherwise be inapplicable. Any conclusion we draw from the model would of course not have to rely on this extension; the time scale for any chaotic behavior to set in must be shorter than one cycle for its physical consequences to be meaningful.

We start with the Einstein-Hilbert action for the gravitational field,

$$S_\mathrm{g}[g] = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R\;, \tag{2.2}$$

and the action

$$S_\mathrm{f}[\Phi, g] = -\tfrac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\left[g^{ab}\nabla_{\!a}\Phi\,\nabla_{\!b}\Phi + (\mu^2 + \tfrac{1}{6}\,R)\,\Phi^2\right] \tag{2.3}$$

for a real, conformally coupled scalar field of mass μ. To write down the Hamiltonian, use the fact that the square root of the determinant of the metric (2.1) and its scalar curvature are, respectively,

$$\sqrt{-g} = a^4\sin^2\!\chi\,\sin\theta\;, \qquad R = 6\left(\frac{\ddot a}{a^3} + \frac{1}{a^2}\right), \tag{2.4}$$

where an overdot denotes a t-derivative, and reparametrize the scalar field (which must be homogeneous for consistency) by $\Phi \mapsto \phi := \sqrt{4\pi G/3}\,a\Phi$. Then we obtain, up to a constant overall factor,

$$H(a,\phi;\pi,p) = \tfrac{1}{2}\left[\,-(\pi^2 + a^2) + (p^2 + \phi^2) + \mu^2a^2\phi^2\right]\;, \tag{2.5}$$

where π and p are the momenta conjugate to a and ϕ, respectively. Here, as usual with general relativistic systems, time reparametrization invariance requires that we impose that all solutions satisfy the constraint

$$H = 0 \,, \tag{2.6}$$

which is just the $(0,0)$-component of the Einstein equation.

The system described by the Hamiltonian (2.5) consists of two harmonic oscillators (of which one is "inverted," as is to be expected from any degree of freedom related to the spatial volume element), coupled through a term proportional to μ^2. For $\mu^2 = 0$, the system is trivially integrable; our main goal is to analyze the effect of the coupling term, at least for small values of μ^2. However, in view of the perturbative treatment to be carried out, the form (2.5) has two disadvantages: the "perturbation" term $\mu^2 a^2 \phi^2$ is not always small, in the sense that for a fixed value of μ^2 it cannot be uniformly bounded for all orbits of the unperturbed system; and the latter system is degenerate, in the sense that the frequencies of both oscillators are constant. We will not attempt to deal directly with the first inconvenience, which at any rate is not really a problem for perturbations of a fixed orbit; we will instead perform a change of variables that will allow us to isolate a different integrable Hamiltonian from H, with enough of the degeneracy removed for us to be able to use the Melnikov method of detecting homoclinic chaos. Regarding the perturbation, it will be sufficient for us to know that it is a well-behaved function on each trajectory.

Following Ref. 1, we first replace the dynamical variables p and ϕ by new variables j and φ, respectively, defined as

$$\phi = \sqrt{\frac{2j}{\omega}}\, \sin\varphi \,, \qquad p = \sqrt{2\,\omega j}\, \cos\varphi \,, \tag{2.7}$$

where $\omega = \sqrt{1+\mu^2 a^2}$ is the instantaneous frequency of the field; thus, this is a first approximation to action-angle variables for the scalar field, which takes into account the coupling term, but not the fact that a is a dynamical variable, and it is a key trick in this treatment of the model. To complete the canonical transformation, we must introduce a new momentum variable conjugate to a, given by

$$P := \pi - \frac{\mu^2 a j}{2\,\omega^2}\, \sin 2\varphi \,, \tag{2.8}$$

and arising from the fact that ω is a-dependent. Now the Hamiltonian in terms of the new variables can be written as a sum $H = \hat{H}_0 + \delta\hat{H}$, of an unperturbed Hamiltonian

$$\hat{H}_0(a, \varphi; P, j) = -\tfrac{1}{2}\,(P^2 + a^2) + j\sqrt{1+\mu^2 a^2} \,, \tag{2.9}$$

which is obviously integrable, since it has the two commuting constants of the motion $\hat{H}_0$ and j, and a perturbation

$$\delta\hat{H}(a,\varphi;P,j) = -\frac{\mu^2 aPj}{2\,(1+\mu^2a^2)}\sin 2\varphi - \left[\frac{\mu^2 aj}{4\,(1+\mu^2a^2)}\right]^2 (1-\cos 4\varphi)\;. \tag{2.10}$$

However, the system (2.9) is not easy to integrate; since we will need to perform explicit calculations, and in particular to write it in terms of action-angle variables, we find it convenient to simplify it somewhat.

In Ref. 1, this simplification was achieved by noticing that for most of the unperturbed motion, the value of ω is very large, and using the $\omega \gg 1$ approximation to $\hat{H}_0$. In addition, because the treatment was a perturbative one in the parameter μ^2, or more precisely $\mu^2 j$, the second term in the Hamiltonian perturbation (2.10) was dropped, since it contains higher powers of μ^2 than the first one. Notice that, implicit in this way of proceeding is the idea that one considers the μ^2 in $\hat{H}_0$ as a *fixed* parameter, while in $\delta\hat{H}$, $\mu^2 =: \varepsilon$ is rendered temporarily independent of the first one for the sake of the perturbative treatment, and allowed to vary in a neighborhood of $\mu^2 = \varepsilon = 0$. (One may be able to set up a perturbation theory for Hamiltonians of the type $H_0(q,p;\varepsilon) + \varepsilon\,\delta H(q,p)$, where $H_0(q,p;\varepsilon)$ is integrable for all ε, but to our knowledge such a theory is not available yet.)

Here, we will follow a different path. Since we are going to expand the perturbation in powers of μ^2 and neglect terms higher than μ^4, we will do the same for $\hat{H}_0$, and retain there only the terms up to order μ^2 while we will put in the perturbation the higher order ones. The new unperturbed system obtained in this way will be explicitly integrable. As the perturbation parameter, we will use μ^2, rather than the dimensionless $\mu^2 j$; this is mainly to avoid having to explain already at this stage what values of the variable j we are interested in. At any rate, strictly speaking we will not have to talk about $\varepsilon = \mu^2$ being "small" (if we did, we might say that $\mu/\mu_{\rm P} \ll 1$, where $\mu_{\rm P}$ is the Planck mass); the perturbative results we will obtain are valid for ε in some interval near zero, and such topological statements are "dimensionless." Expanding $\sqrt{1+\mu^2a^2} = 1 + \frac{1}{2}\,\mu^2a^2 - \frac{1}{4}\,\mu^4a^4 + o(\mu^4)$, we obtain a new breakup of the Hamiltonian into

$$H = H_0 + \delta H\;, \tag{2.11}$$

where the unperturbed part is now

$$H_0(a,\varphi;P,j) = -\tfrac{1}{2}\,(P^2+a^2) + j\,(1+\tfrac{1}{2}\,\mu^2a^2)\;, \tag{2.12}$$

and the perturbation differs from (2.10) by terms proportional to μ^4a^4 and of higher order in μ^2, namely

$$\delta H = -\tfrac{1}{2}\,\mu^2aPj\,\sin 2\varphi + \mu^4\left[\tfrac{1}{2}\,a^3Pj\,\sin 2\varphi - \tfrac{1}{4}\,a^4j - \tfrac{1}{16}\,a^2j^2\,(1-\cos 4\varphi)\right] + o(\mu^4)\;. \tag{2.13}$$

One of the action variables of the new unperturbed system is j itself; to find the other one, we compute the integral $k := (2\pi)^{-1} \int P\, \mathrm{d}a$, along the trajectory specified by the values of the conserved quantities j and $H_0 = \text{const} =: h_0$ (ultimately, we will be interested only in trajectories near $h_0 = 0$, in order to satisfy the constraint $H_0 + \delta H = 0$, but for now we keep the treatment more general), for which

$$P(a) = \pm \sqrt{2\,(j - h_0) - (1 - \mu^2 j)\, a^2}\,. \tag{2.14}$$

The integration yields

$$k = \frac{j - h_0}{\sqrt{1 - \mu^2 j}}\,, \tag{2.15}$$

and the angle variables canonically conjugate to k and j, respectively, are given by

$$\theta = \arctan\left(\sqrt{1 - \mu^2 j}\, a/P\right) \tag{2.16}$$

$$\delta = \varphi - \frac{\mu^2 a P}{4\,(1 - \mu^2 j)}\,, \tag{2.17}$$

as can be checked by direct computation of the Poisson brackets. Notice that μ^2 here is not the infinitesimal ε, but we do want to consider it as being small, so for j not too big $1 - \mu^2 j > 0$. With these new variables we can now write down our final expression for the unperturbed Hamiltonian,

$$H_0(k, j) = j - k\,\sqrt{1 - \mu^2 j}\,. \tag{2.18}$$

Choosing a value for h_0 will then impose a relationship between j and k; in particular, for $h_0 \approx 0$, since j is positive—see (2.12)—k must be positive as well. For the perturbation, we invert the transformation (2.15)–(2.17), and substitute in (2.13), which, keeping only the lowest order terms in $\varepsilon = \mu^2$, gives

$$\begin{aligned}\delta H = {}& \tfrac{1}{4}\, \varepsilon\, kj \Big[\cos(2\theta + 2\delta) - \cos(2\theta - 2\delta)\Big] + \tfrac{1}{4}\, \varepsilon^2 \Big[-\tfrac{3}{2}\, k^2 j - \tfrac{1}{4}\, kj^2 + \\ & + (2\, k^2 j + \tfrac{1}{4}\, kj^2) \cos 2\theta - \tfrac{1}{2}\, k^2 j \cos 4\theta - \tfrac{1}{2}\, k^2 j \cos(2\delta - \pi/2) + \tfrac{1}{4}\, kj^2 \cos 4\delta + \\ & + k^2 j \cos(2\theta - 2\delta) - k^2 j \cos(2\theta + 2\delta) - \tfrac{1}{8}\, kj^2 \cos(2\theta + 4\delta) - \tfrac{1}{8}\, kj^2 \cos(2\theta - 4\delta) + \\ & + \tfrac{\sqrt{5}}{4}\, k^2 j \cos(4\theta + 2\delta + \psi) - \tfrac{\sqrt{5}}{4}\, k^2 j \cos(4\theta - 2\delta + \psi)\Big] + O(\varepsilon^3)\,,\end{aligned} \tag{2.19}$$

where $\psi := -\arcsin(1/\sqrt{5})$, and we have chosen to write the various terms in the form that will be most useful in the following. This completes the setup for the perturbative treatment of the model. The dynamics of the unperturbed Hamiltonian H_0 is trivial, and gives a conditionally periodic motion $\theta = \theta_0 + \omega_k t$, $\delta = \delta_0 + \omega_j t$, with frequencies given by

$$\begin{aligned}\omega_k &= \frac{\partial H_0}{\partial k} = -\sqrt{1 - \mu^2 j} \\ \omega_j &= \frac{\partial H_0}{\partial j} = \frac{\mu^2 k + 2\,\sqrt{1 - \mu^2 j}}{2\,\sqrt{1 - \mu^2 j}}\,.\end{aligned} \tag{2.20}$$

In the next section we will start to take a look at the perturbed dynamics.

III. Effect of the resonant terms in the perturbation

The effect of the perturbation δH is felt in particular on the rational tori of the unperturbed dynamics, the ones where the motion becomes periodic, because the frequencies (2.20) associated with the two degrees of freedom are related by the resonance condition

$$n_0\, \omega_k + m_0\, \omega_j = 0 \;, \tag{3.1}$$

for some integer numbers n_0 and m_0. Using the explicit form of the frequencies in the resonance condition, we obtain a relationship between the action variables k and j, namely that

$$\frac{2\,(1-\mu^2 j)}{\mu^2 k + 2\,\sqrt{1-\mu^2 j}} = \frac{m_0}{n_0} \tag{3.2}$$

must be a rational number. The latter equation can be expressed as

$$\mu^2 k = 2\,\frac{n_0}{m_0}\,(1-\mu^2 j) - 2\,\sqrt{1-\mu^2 j}\;, \tag{3.3}$$

and, since k must be positive in order for $h_0 \approx 0$, we see that a necessary condition is that n_0 and m_0 have the same sign—we can take them to be positive—, with n_0 strictly greater than m_0.

To find out which of the resonant tori of the unperturbed system are affected by δH, we must consider the perturbation (2.19) as a Fourier series with respect to the angular variables θ and δ, of the form

$$\begin{aligned} \delta H(\theta,\delta;k,j) &= \sum_{i=1}^{\infty} \delta H^{(i)}(\theta,\delta;k,j) \\ \delta H^{(i)}(\theta,\delta;k,j) &= \varepsilon^i \sum_{n,m} V^{(i)}_{nm}(k,j)\, \cos(n\theta + m\delta + \psi^{(i)}_{nm})\;, \end{aligned} \tag{3.4}$$

and check which of the Fourier coefficients $V^{(1)}_{nm}(k,j)$ are non-vanishing. The order ε terms in (2.19) are

$$\delta H^{(1)} = \tfrac{1}{4}\,\varepsilon\, kj \Big[\cos(2\theta+2\delta) - \cos(2\theta-2\delta)\Big]. \tag{3.5}$$

Therefore, to order μ^2 in the perturbation, the resonant tori which are broken by the perturbation are those for which

$$n_0 = \pm\, m_0 = 2\;.$$

However, neither of these pairs satisfies the conditions given below (3.3), which means that these resonant tori are not present in the physically relevant region near $h_0 = 0$.

This forces us to consider the Fourier components of order ε^2 in the perturbation (2.19). In order to look at their effect on the dynamics, however, we must somehow consider them as perturbations of a system which includes the order ε terms. The dynamical behavior of such a system is qualitatively similar to that of the unperturbed one in the region near $h_0 = 0$, because H_0 has no tori there which resonate with the order ε terms, and to set up the system we can use the normal techniques for non-resonant perturbations of integrable systems.

The method (see, e.g., Ref. 16, §5.10) consists in performing an ε-dependent canonical transformation to new coordinates $(\theta', \delta'; k', j')$, in which the new Hamiltonian has no perturbation terms of order ε, and is obtained by using $k'\theta + j'\delta + \Phi(\theta, \delta; k', j')$ as generating function, where

$$\Phi(\theta, \delta; k', j') = -\varepsilon \sum_{n,m} \frac{V^{(1)}_{nm}(k', j')}{n\,\omega_k(k', j') + m\,\omega_j(k', j')} \sin(n\theta + m\delta + \psi_{nm}) \,. \tag{3.6}$$

While the new canonical coordinates cannot be explicitly expressed in terms of the old ones, they differ by a series of powers of ε, and for our purposes it will be sufficient to calculate the term of order ε. A calculation shows that the new unperturbed Hamiltonian $H'_0(k', j') = H_0(k', j')$, i.e., it is given by the same function (2.18) as in the previous variables, while the perturbation is now of the form

$$\begin{aligned}
\delta H' = \varepsilon^2 \Big[& V'^{(2)}_{00}(k', j') + V'^{(2)}_{20}(k', j') \cos 2\theta' + V'^{(2)}_{02}(k', j') \cos(2\delta' - \pi/2) + \\
& + V'^{(2)}_{40}(k', j') \cos 4\theta' + V'^{(2)}_{04}(k', j') \cos 4\delta' + \\
& + V'^{(2)}_{22}(k', j') \cos(2\theta' + 2\delta') + V'^{(2)}_{2-2}(k', j') \cos(2\theta' - 2\delta') + \\
& + V'^{(2)}_{24}(k', j') \cos(2\theta' + 4\delta') + V'^{(2)}_{2-4}(k', j') \cos(2\theta' - 4\delta') + \\
& + V'^{(2)}_{42}(k', j') \cos(4\theta' + 2\delta' + \psi) + V'^{(2)}_{4-2}(k', j') \cos(4\theta' - 2\delta' + \psi) + \\
& + V'^{(2)}_{44}(k', j') \cos(4\theta' + 4\delta') + V'^{(2)}_{4-4}(k', j') \cos(4\theta' - 4\delta') \Big] + O(\varepsilon^3) \,.
\end{aligned} \tag{3.7}$$

Since the unperturbed Hamiltonian is the same as before, the condition for a resonance at $h_0 \approx 0$ is still $n_0 > m_0 > 0$, and the only term in (3.7) satisfying this condition has

$$n_0 = 4 \,, \quad m_0 = 2 \,,$$

in which case

$$\mu^2 k_0 = 2\sqrt{1 - \mu^2 j_0}\,\big(2\sqrt{1 - \mu^2 j_0} - 1\big), \tag{3.8}$$

is positive if $\mu^2 j_0 < 3/4$. We thus choose to study the motion near the resonant torus with action variables (j_0, k_0) characterized by the integer numbers $(n_0, m_0) = (4, 2)$, and for which the perturbation coefficient is

$$V'^{(2)}_{42}(k', j') = V^{(2)}_{42}(k', j') = \tfrac{\sqrt{5}}{16}\, k'^2 j'. \tag{3.9}$$

(It is not necessary for us to write down explicitly all the functions $V'^{(2)}_{mn}(k', j')$ here, but some of them coincide with the $V^{(2)}_{mn}(k', j')$ in (2.19).) As far as this local dynamics is concerned, then, we could have used directly the Hamiltonian $H_0(k, j)$ with perturbation $\delta H^{(2)}(\theta, \delta; k, j)$ as in (2.19), without worrying about the presence of the ε term. From now on, we will drop the primes on the variables just introduced.

Since the perturbation with coefficient (3.9) is resonant, it does change the local dynamics qualitatively near the chosen torus. Following the standard procedure for such a case (see, e.g. Ref. 9), we go over to a set of canonical variables $(\gamma, \delta; K, J)$ adapted to the resonant torus, chosen so that one of the momenta will still be a constant of the motion under the resonant term in the perturbation:

$$\begin{aligned} K &:= \frac{k - k_0}{n_0} = \tfrac{1}{4}\,(k - k_0) \qquad\qquad & \gamma &:= n_0\,\theta + m_0\,\delta + \psi_0 = 4\,\theta + 2\,\delta + \psi_0 \\ J &:= -\frac{m_0}{n_0}\,k + j = -\tfrac{1}{2}\,k + j \end{aligned} \tag{3.10}$$

where ψ_0 is some arbitrary, fixed angle.

If we suppose that $K \ll 1$ is a small increment of the variable k around the resonant value k_0, then we can expand the Hamiltonian, written in the new variables, in powers of K, and study the dynamics generated by the leading terms. We begin with the resonant part of the perturbation,

$$\begin{aligned} \delta H^{(2)} &= \varepsilon^2\, V^{(2)}_{42}(k, j)\cos(4\theta + 2\delta + \psi) + (\text{terms with different } n \text{ and } m) \\ &= \tfrac{\sqrt{5}}{16}\,\varepsilon^2 k_0^2\, j\, \cos(\gamma + \psi - \psi_0) + (\text{terms with different } n \text{ and } m)\ . \end{aligned} \tag{3.11}$$

Here, j is to be thought of as $j(K = 0, J)$, with J arbitrary. Since this term in the perturbation depends only on γ and not on δ, J is still a constant of the motion; for simplicity we fix its value at the resonant one, J_0. Then for the unperturbed Hamiltonian we obtain

$$H_0(K, J_0) = H_0(k_0, j_0) + \tfrac{1}{2}\,\Omega\, K^2 + O(K^3)\ , \tag{3.12}$$

where

$$\Omega := \frac{\partial^2 H_0}{\partial k^2}\, n_0^2 + 2\,\frac{\partial^2 H_0}{\partial j \partial k}\, n_0 m_0 + \frac{\partial^2 H_0}{\partial j^2}\, m_0^2 = \frac{8\,\mu^2}{\sqrt{1 - \mu^2 j_0}} + \frac{\mu^4 k_0}{(1 - \mu^2 j_0)^{3/2}}\ . \tag{3.13}$$

So, to the lowest order in K and ε, the K dynamics near the resonant torus is generated by

$$H_{\rm loc}(\gamma, K) = H_0(k_0, j_0) + \tfrac{1}{2}\,\Omega\,K^2 + \tfrac{\sqrt{5}}{16}\,\varepsilon^2 k_0^2\, j_0 \cos(\gamma + \psi - \psi_0)\;, \tag{3.14}$$

which is the Hamiltonian of a well-known system, the non-linear pendulum.

Let us now find the homoclinic orbits. If we call $h_0 = H_0(k_0, j_0)$, as before, and fix some value $H_{\rm loc} = h_{\rm loc}$ for the local Hamiltonian, we can compute from Eq. (3.14)

$$K = \pm \left\{ \frac{2}{\Omega} \Big[h_{\rm loc} - h_0 - \tfrac{\sqrt{5}}{16}\,\varepsilon^2 k_0^2\, j_0 \cos(\gamma + \psi - \psi_0) \Big] \right\}^{1/2} .$$

To simplify calculations, we set $\psi_0 = \psi + \pi$. Then, for the homoclinic orbit, at the maximum $\gamma = \pi$ of the potential we must have $K = 0$, from which we find that $h_{\rm loc} - h_0 = \frac{\sqrt{5}}{16}\,\varepsilon^2 k_0^2\, j_0$, and

$$K = \pm \left\{ \tfrac{\sqrt{5}}{8}\,\Omega^{-1} \varepsilon^2 k_0^2\, j_0\, (1 + \cos\gamma) \right\}^{1/2} = \pm \left(\tfrac{\sqrt{5}}{4}\,\Omega^{-1} \varepsilon^2 k_0^2\, j_0 \right)^{1/2} \cos\frac{\gamma}{2}\;. \tag{3.15}$$

Finally, we can compute the evolution of γ by integrating the Hamilton equation $\dot\gamma = \partial H_{\rm loc}/\partial K = \Omega\, K$, and we obtain

$$\gamma(t) = 4 \arctan \exp \left\{ \left(\tfrac{\sqrt{5}}{16} \varepsilon^2 \Omega\, k_0^2\, j_0 \right)^{1/2} t \right\} - \pi\;. \tag{3.16}$$

We will use all of these results in the next section.

IV. The Melnikov method

We are now going to study the effect of the other terms of the perturbation

$$\begin{aligned} \delta H'^{(2)}(\gamma, \delta; K, J) &= \varepsilon^2 \sum_{nm} V'^{(2)}_{nm}(k, j) \cos(n\theta + m\delta + \psi_{nm}) \\ &= \varepsilon^2 \sum_{nm} V'^{(2)}_{nm}(k, j) \cos\left(\frac{n}{4}\,\gamma + \frac{2m - n}{2}\,\delta - \frac{n}{4}\,\psi_0 + \psi_{nm} \right), \end{aligned} \tag{4.1}$$

in particular of the terms with $(n, m) \neq (n_0, m_0)$, on the dynamics of the integable system with Hamiltonian $H_{\rm loc}$, near the homoclinic orbit. We will use the Melnikov method to show that a stochastic layer forms in the vicinity of this destroyed separatrix, which acts as a seed for chaos.

We thus want to show[13] that the Melnikov function fo the local K dynamics has a contribution

$$M_{nm}(\psi_{nm}) := \int_{\rm h.o.} {\rm d}t\, \{H_{\rm loc}, \delta H'^{(2)}_{nm}\} \tag{4.2}$$

which is an oscillating function of ψ_{nm} and has therefore isolated, transverse zeroes, for some values of (n, m); here, the integral in (4.2) is taken over the homoclinic orbit, given

by $\gamma(t)$ in (3.16), $\delta(t) = \omega_j t + O(\varepsilon^2)$ from $\dot\delta = \partial H/\partial j'$, $k = k_0 + 4K = k_0 + O(\varepsilon)$, and $j = j_0 + O(\varepsilon)$.

In the calculation of the Melnikov integral we will keep only the leading order terms in ε and K. Then, approximating the Poisson bracket

$$\begin{aligned}\{H_{\rm loc}, \delta H'^{(2)}_{nm}\} &\approx \Big\{\tfrac12\,\Omega K^2, \varepsilon^2 V'^{(2)}_{nm}\cos\Big(\frac{n}{4}\,\gamma + \frac{2m-n}{2}\,\delta - \frac{n}{4}\,\psi_0 + \psi_{nm}\Big)\Big\}\\ &= \tfrac14\,\varepsilon^2\Omega K n\, V'^{(2)}_{nm}(k_0, j_0)\sin\Big(\frac{n}{4}\,\gamma + \frac{2m-n}{2}\,\delta - \frac{n}{4}\,\psi_0 + \psi_{nm}\Big), \end{aligned}\tag{4.3}$$

and using $\dot\gamma = \Omega K$, we get

$$\begin{aligned}M_{nm}(\psi_{nm}) &= \int_{-\pi}^{\pi} {\rm d}\gamma\,\dot\gamma^{-1}\,\{H_{\rm loc}, \delta H'^{(2)}_{nm}\}\\ &= \tfrac14\,\varepsilon^2 n\, V'^{(2)}_{nm}(k_0, j_0)\big(A_{nm}\sin\psi'_{nm} + B_{nm}\cos\psi'_{nm}\big), \end{aligned}\tag{4.4}$$

where we have defined $\psi'_{nm} := \psi_{nm} - \frac{n}{4}\psi_0$, and

$$\begin{aligned}A_{nm} &:= \int_{-\pi}^{\pi} {\rm d}\gamma\,\cos\Big(\frac{n}{4}\,\gamma + \frac{2m-n}{2}\,\delta(\gamma)\Big)\\ B_{nm} &:= \int_{-\pi}^{\pi} {\rm d}\gamma\,\sin\Big(\frac{n}{4}\,\gamma + \frac{2m-n}{2}\,\delta(\gamma)\Big), \end{aligned}\tag{4.5}$$

and $\delta(\gamma)$ is obtained from (3.16) and $\delta \approx \omega_j t$,

$$\delta(\gamma) = \eta\,\ln\,\tan\frac{\gamma+\pi}{4}\,, \qquad \eta := \omega_j\,\Big(\tfrac{\sqrt5}{16}\varepsilon^2\Omega\,k_0^2\,j_0\Big)^{-1/2}\,. \tag{4.6}$$

The function M_{nm} in (4.4) is clearly oscillating whenever $nV'^{(2)}_{nm}A_{nm}$ and/or $nV'^{(2)}_{nm}B_{nm}$ are not zero. We thus have to show that, for some (n, m) in the set $\{(2, 0)$, $(2, \pm2)$, $(2, \pm4)$, $(4, 0)$, $(4, -2)$, $(4, \pm4)\}$, A_{nm} or B_{nm} don't vanish.

Perhaps surprisingly, the integrals in (4.5) can be calculated analytically, at least for half of the cases of interest, the ones with $n = 2$. To do this, notice that A_{nm} and B_{nm} are, respectively, the real and imaginary parts of the complex function

$$\begin{aligned}Z_{nm} &:= \int_{-\pi}^{\pi} {\rm d}\gamma\,{\rm e}^{{\rm i}n\gamma/4}\,\Big(\tan\frac{\gamma+\pi}{4}\Big)^{{\rm i}(2m-n)\eta/2}\\ &= 4\,{\rm e}^{-{\rm i}n\pi/4}\int_0^{\pi/2}{\rm d}x\,{\rm e}^{{\rm i}nx}(\tan x)^{{\rm i}(2m-n)\eta/2}, \end{aligned}\tag{4.7}$$

where we have defined $x := \frac14\,(\gamma+\pi)$. Therefore, for $n = 2$, we have[17]

$$\begin{aligned}Z_{2m} &= -4{\rm i}\Big[\int_0^{\pi/2}{\rm d}x\,\cos 2x\,\tan^{{\rm i}(m-1)\eta}x + {\rm i}\int_0^{\pi/2}{\rm d}x\,\sin 2x\,\tan^{{\rm i}(m-1)\eta}x\Big]\\ &= -4{\rm i}\left[-\frac{{\rm i}(m-1)\eta\pi}{2}\,\sec\frac{{\rm i}(m-1)\eta\pi}{2} + \frac{{\rm i}(m-1)\eta\pi}{2}\,{\rm cosec}\,\frac{{\rm i}(m-1)\eta\pi}{2}\right]\\ &= -2(m-1)\eta\pi\left[{\rm sech}\,\frac{(m-1)\eta\pi}{2} + {\rm i}\,{\rm cosech}\,\frac{(m-1)\eta\pi}{2}\right]; \end{aligned}\tag{4.8}$$

in other words,

$$A_{2m} = -2(m-1)\eta\pi \,\mathrm{sech}\, \frac{(m-1)\eta\pi}{2}\ , \quad B_{2m} = -2(m-1)\eta\pi \,\mathrm{cosech}\, \frac{(m-1)\eta\pi}{2}\ , \quad (4.9)$$

which don't vanish for any of the values of m listed above.

V. Conclusions

We have shown analytically that a RW universe, filled with a massive conformally coupled scalar field, has a chaotic behaviour, at least for a sufficiently small value of the mass, confirming the numerical results of Ref. 1. This is just the first step in our program to analyze the dynamics of RW models with various types of matter content, in order to relate this to numerical results, and make more precise statements about physical consequences. Among the purely theoretical aspects of this work, on the one hand, some possible generalizations to cosmological models differing by the addition of a cosmological constant and/or other parameters are being currently studied,[18] as well as the possibility of adding a second matter field; on the other hand, we are working on extending the results obtained for the model in this paper to an estimation of the size of the stochastic region in phase space and of the time scale for the manifestation of the chaos we predict. The latter point is an important one since, as we mentioned in §2, this time scale must be smaller than the lifetime of the universe in our model. Although the simulations in Ref. 1 covered many cycles for the universe, it was stated there that chaos would show up on much smaller time scales, and therefore has, in principle, observable consequences; in Ref. 7, however, the authors argued qualitatively that they expect the time scale to be much larger than the lifetime of the universe, and used this to motivate the use of two different scalar fields in their model. We believe that the issue should be resolved by an actual, quantitative estimate of the time scale.

We will conclude with some comments on possible avenues for future research, concerning other aspects of the relationship between chaos and the physics of the early universe. At a classical level already, chaos opens up ways of explaining the origin of the arrow of time,[19] and in our cosmological setting therefore, of improving our knowledge of cosmological statistical mechanics and thermodynamics. Furthermore, just as non-integrability and dynamical instability are, at the classical level, the causes of chaos, they are also, at the quantum level, the causes of the instability of quantum states and particles; one therefore expects, for example, classical chaos and semiclassical particle production to be related.[20] At the semiclassical level,[21] one can prove the presence of decoherence and correlations, which cause the transition to the classical regime; stable and unstable scalar field quanta

are created during this process. The study of this passage, that leads from particle creation to chaos is, therefore, imperative. In a model very similar to ours,[22] a Lyapunov variable was used to define a growing entropy in the universe; this could be adapted, with minor changes, to the present model, and looks primising since, for example, growth of entropy, particle creation, decoherence and isotropization (a tendency to some kind of equilibrium) are related in semiclassical cosmological models.[23]

The relevance of chaos in the transition to the semiclassical limit of quantum cosmology, for RW models, was studied in Ref. 24. At the complete quantum level, let us recall only that RW models have been used to study the relation between the cosmological and thermodynamical arrows of time;[25] but many other results related to these subjects can be found in the literature. Ultimately, a very interesting goal for future work, and possibly a formidable task even in our simple RW model with chaotic behaviour, is to relate all these fundamental features of the universe in a unified theoretical framework.

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